Kadlu Sound Propagation

Oliver S. Kirsebom

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1 The PE Equation

Kadlu uses a Parabolic Equation (PE) approch for solving the wave equation (Jensen, Ch. 6). This yields solutions that are valid in the farfield (distance from source considerable greater than wavelength) for environments that exhibit "weak" range dependence. More specifically, Kadlu implements a numerical solution to the PE equation introduced by Thomson and Chapman,

$$\frac{\partial \psi}{\partial r} = ik_0 \left(n - 2 + \left[1 + k_0^{-2} \frac{\partial^2}{\partial z^2} \right]^{1/2} \right) \psi \tag{1}$$

which is said to have "good wide-angled behavior for realistic ocean acoustic environments with moderate changes in the refraction index", $n \equiv c_0/c$. Rearranging and introducing $A \equiv ik_0(n-1)$ and $B \equiv ik_0[-1 + (1 + k_0^{-2} \frac{\partial^2}{\partial z^2})^{1/2}]$, this can be written in the compact form,

$$\frac{\partial \psi}{\partial r} = (A + B)\psi \tag{2}$$

2 Split-Step Fourier Solution

Following Jensen Eq. (6.123), we approximate the solution with,

$$\psi(r + \Delta r) \;\approx\; e^{\frac{B}{2}\Delta r} e^{A \Delta r} e^{\frac{B}{2}\Delta r} \psi(r) \tag{3}$$

Finally, using $\mathcal{F}$ to denote the fourier transform $z \to k_z$ and using the correspondence $\frac{\partial^2}{\partial z^2} \to -k_z^2$ (Jensen Eq. (6.87)), we obtain the split-step Fourier formula,

$$\mathcal{F}\psi(r + \Delta r, z) \;\approx\; U_{\mathrm{D}}(\tfrac{1}{2}\Delta r)\, \mathcal{F}\, U_{\mathrm{R}}(\Delta r)\, \mathcal{F}^{-1}\, U_{\mathrm{D}}(\tfrac{1}{2}\Delta r)\, \mathcal{F}\psi(r, z)\;, \tag{4}$$

where U_{D} and U_{R} are the *diffractive* and *refractive* propagation matrices, respectively,

$$\begin{aligned} U_{\mathrm{D}}(x) \;&=\; e^{i\left[\left(k_0^2 + k_z^2\right)^{1/2} - k_0\right]x}\;, \\ U_{\mathrm{R}}(x) \;&=\; e^{ik_0(n-1)x} \end{aligned} \tag{5}$$

3 Computational domain

Following Jensen Sec. 6.5.3, we implement the split-step Fourier algorithm on a uniform grid $(\Delta r, \Delta z)$. Following Jensen, we adopt a default grid size of,

$$\Delta z = \lambda/2\ , \quad \Delta r = 2\Delta z\ , \quad \lambda \equiv c_0/f = 2\pi/k_0\ , \tag{6}$$

where $c_0 = 1,500$ m/s is the reference sound speed in water. (The option is provided for the user to specify a finer/coarser grid as needed.) The water surface ($z = 0$) is treated as a pressure-release surface, requiring $\psi(r, 0) = 0$. At the bottom, we terminate the physical solution domain by an artificial absorption layer of uniform thickness and a complex index of refraction of the form,

$$n^2 = n_b^2 + i\alpha e^{-(|z|-z_{\max})^2/D^2}\ , \tag{7}$$

where we adopt $\alpha = 1/(\pi \log_{10} e) \approx 0.733$, $D = (z_{\max} - H)/3$, and $z_{\max} = \frac{4}{3}H$. We determine the depth at which the physical domain is terminated, H, from the requirement that the real bottom should have a thickness of at least several wavelengths. Thus, we take,

$$H = \max z_b + 3\lambda\ , \tag{8}$$

where $\max z_b$ is the maximum seafloor depth in the domain.

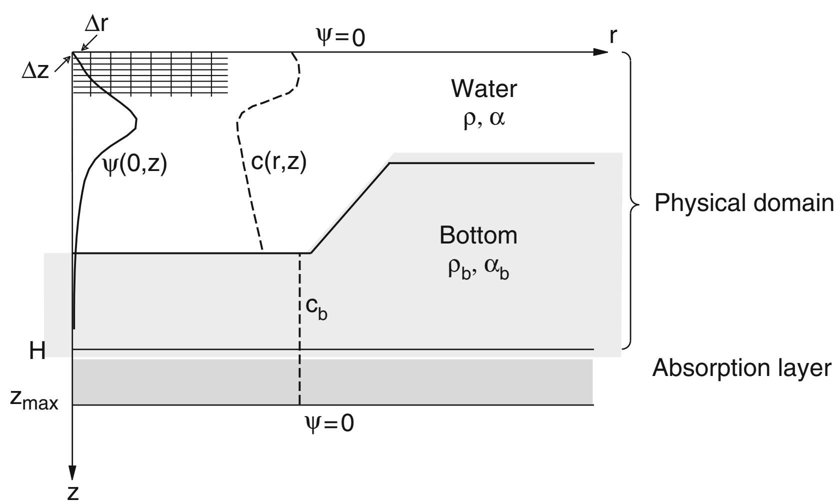

Figure 1: Shematic of PE solution domain (adapted from Jensen).

4 Water-Bottom Interface

For realistic treatment of bottom effects on sound propagation, it is important to include density changes at the water-bottom interface. We follow the approach described in Jensen Sec. 6.5.4. The refractive index, n, is replaced with the *effective* index of refraction,

$$\widetilde{n}^2 = n^2 + \frac{1}{2k_0^2}\left[\frac{1}{\rho}\nabla^2\rho - \frac{3}{2\rho^2}(\nabla\rho)^2\right] , \tag{9}$$

and the displacement potential ψ in Eqs. (1)–(4) is replaced with $\widetilde{\psi} = \psi/\sqrt{\rho}$. We remove density discontinuities at the water-bottom interface by introducing a smoothing function of the form,

$$\rho(z) = \rho + \tfrac{1}{2}(\rho_b - \rho)\tanh\chi , \tag{10}$$

where $\chi \equiv \frac{|z|-z_b}{L}$, z_b being the seafloor depth and L the distance over which the density changes from ρ to ρ_b. We adopt $L = \pi/k_0 = \lambda/2$, close to the value of $L = 2/k_0$ suggested by Jensen. By taking the gradient of Eq. (10), we further obtain,

$$|\nabla\rho| = \frac{1}{2L}(\rho_b - \rho)\,\mathrm{sech}^2\chi\left[1 + (\nabla z_b)^2\right]^{1/2} , \tag{11}$$

$$\nabla^2\rho \approx -\frac{1}{L^2}(\rho_b - \rho)\,\mathrm{sech}^2\chi\tanh\chi\left[1 + (\nabla z_b)^2\right] \tag{12}$$

where $(\nabla z_b)^2 = \left(\frac{\partial z_b}{\partial r}\right)^2 = \cos^2\phi\left(\frac{\partial z_b}{\partial x}\right)^2 + \sin^2\phi\left(\frac{\partial z_b}{\partial y}\right)^2$. Moreover, in the second equation, we have neglected the curvature of the seafloor, i.e., $\nabla^2 z_b \approx 0$. We note that Kadlu assumes a single bottom layer, although it would be straightforward to generalize the implementation to handle several layers.

5 Volume Attenuation

We ignore volume attenuation in the water column, but include volume attenuation in the bottom layer by subtracting a complex term from the sound speed, c_b. The complex term is computed as the largest, real root of the polynomial $\beta x^2 - x + \beta c_b^2 = 0$ fulfilling $0 \leq x < c_b$. Here $\beta = \alpha_b^{(\lambda)}/(40\pi c_b \log_{10} e)$ with $\alpha_b^{(\lambda)}$ being the attenuation coefficient in units of dB/λ. For typical values of $\alpha_b^{(\lambda)}$, this leads to,

$$c \rightarrow c_b(1 - i\beta c_b) \tag{13}$$

$$n_b^2 \rightarrow n_b^2(1 + i2\beta c_b) \tag{14}$$

6 Starter

We use the Thomson starter field, as defined in Jensen Sec. 6.4.2.3,

$$\psi(0,k_z) \;=\; \begin{cases} \frac{e^{-i\pi/4}}{\Delta z}\,(8\pi)^{1/2}\left(k_0^2-k_z^2\right)^{-1/4}\sin k_z z_s \;, & |k_z| < k_0\sin\theta_1 \\ 0\;, & |k_z| \geq k_0\sin\theta_1 \end{cases} \tag{15}$$

where $\psi(0,k_z) \equiv \mathcal{F}\psi(0,z)$, while z_s and θ_1 are the depth and half-beamwidth of the source, respectively, and Δz is the vertical grid spacing. Note that the prefactor $\frac{e^{-i\pi/4}}{\Delta z}$ does not appear in the formula given in Jensen Sec. 6.4.2.3.